Ex.22　解: (1) 由条件

$$
\begin{aligned}
A\alpha_1 &= \alpha_1+\alpha_2+\alpha_3,\\
A\alpha_2 &= 2\alpha_2+\alpha_3,\\
A\alpha_3 &= 2\alpha_2+3\alpha_3
\end{aligned}
$$

得

$$A(\alpha_1,\alpha_2,\alpha_3)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&0&0\\1&2&2\\1&1&3\end{pmatrix}.$$

(2) 记

$$Q=(\alpha_1,\alpha_2,\alpha_3),\quad B=\begin{pmatrix}1&0&0\\1&2&2\\1&1&3\end{pmatrix}.$$

则$A=QBQ^{-1}$, 即A 与B 相似, 所以, A 与B 有相同特征值. 由

$$|\lambda E-B|=\begin{vmatrix}\lambda-1&0&0\\-1&\lambda-2&-2\\-1&-1&\lambda-3\end{vmatrix}=(\lambda-1)\begin{vmatrix}\lambda-2&-2\\-1&\lambda-3\end{vmatrix}=(\lambda-1)^2(\lambda-4).$$

所以, 矩阵B 的特征值为$1,1,4$, 即矩阵A 的特征值为$1,1,4$.

(3) 当$\lambda=1$ 时, 解齐次线性方程组$(E-B)x=0$. 由

$$E-B=\begin{pmatrix}0&0&0\\-1&-1&-2\\-1&-1&-2\end{pmatrix}\to\begin{pmatrix}1&1&2\\0&0&0\\0&0&0\end{pmatrix}$$

得基础解系

$$\xi_1=\begin{pmatrix}-1\\1\\0\end{pmatrix},\quad \xi_2=\begin{pmatrix}-2\\0\\1\end{pmatrix}.$$

当$\lambda=4$ 时, 解齐次线性方程组$(4E-B)x=0$. 由

$$4E-B=\begin{pmatrix}3&0&0\\-1&2&-2\\-1&-1&1\end{pmatrix}\to\begin{pmatrix}1&0&0\\0&1&-1\\0&0&0\end{pmatrix}$$

得基础解系

$$\xi_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

记矩阵

$$S = (\xi_1, \xi_2, \xi_3) = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

则$B = S\Lambda S^{-1}$, 所以,

$$A = QBQ^{-1} = QS\Lambda S^{-1}Q^{-1} = (QS)\Lambda(QS)^{-1}.$$

令$P = QS$ 得$P^{-1}AP = \Lambda$, 其中,

$$P = QS = (\alpha_1, \alpha_2, \alpha_3)\begin{pmatrix} -1 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = (-\alpha_1 + \alpha_2, -2\alpha_1 + \alpha_3, \alpha_2 + \alpha_3).$$